Dynamics 4

Learning Module 7

Impact on flexible structures



THE UNIVERSITY *of* EDINBURGH
School of Engineering

DYNAMICS 4

LM7 GUIDE 26 27

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Learning Module 7 | Impact on flexible structures

Estimated time: 13 hours Complete module before: 4 December

1 Start by reading the Course Notes: For this module you should read **Sections 4.1 and 4.2.** Take hand-written notes as you read.

2 Attend the lectures and tutorial sessions and, if you need, watch the lecture recordings later. This will help consolidate what you learned. Keep taking notes.

3 Attempt to solve **Questions 7.1 to 7.8** in **Exercise Sheet 7**. Compare your solutions to the numerical solutions provided at the end of each question. Write down any questions you might have.

4 Compare your solutions to the worked-out example solutions to **Questions 7.1 to 7.8**. Discuss any questions you might have in the tutorial sessions. Arrange to meet with your lecturer if you need further clarifications.

5 Be active in the Learn discussion board thread for this Learning Module, which is labeled **impact_on_structures**. You can post your own questions and help clarify or answer the questions of others.



In this module

Dynamics of continuous systems
Impact on flexible structures
Single/multiple degree-of-freedom approximation
Transverse impact on elastic beams
Modified dynamic structural response

Learning outcomes

Derive equations of motion for dynamic systems from energy-based approaches (e.g. Lagrange's method)

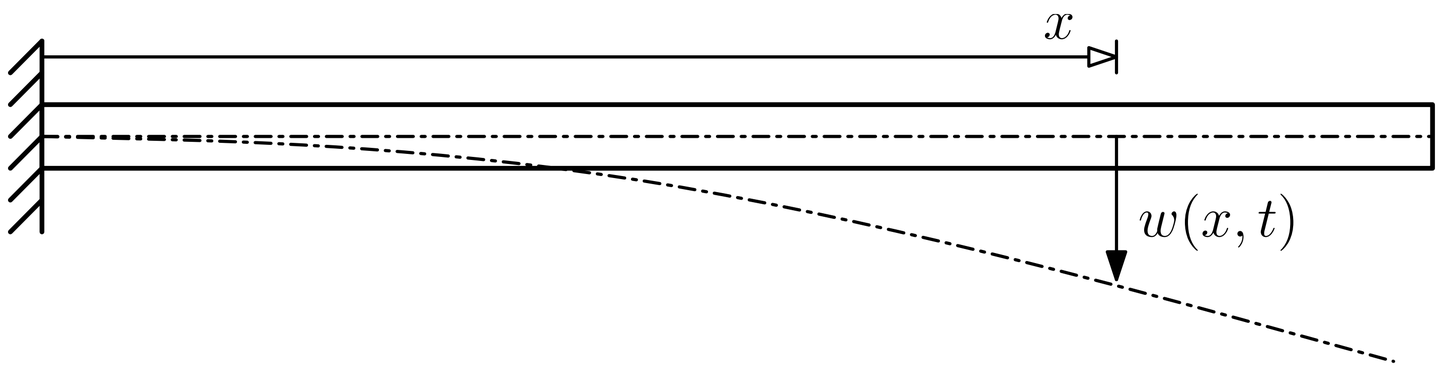
Know the common wave equations for basic structural elements and be able to use these to better understand the response of simple structural elements

This is a study and learning guide to Learning Module 7. You can find the corresponding lecture recordings and slides on Learn. This module has a dedicated thread in the discussion board on Learn.

Please note that the course time and the recommended *before dates*, are estimated and may differ from person to person. You should not see these as targets but instead as a way to help you plan your semester and the preparation for exams. The estimated time includes going through all the course materials, interaction in the discussion boards and self-study time.

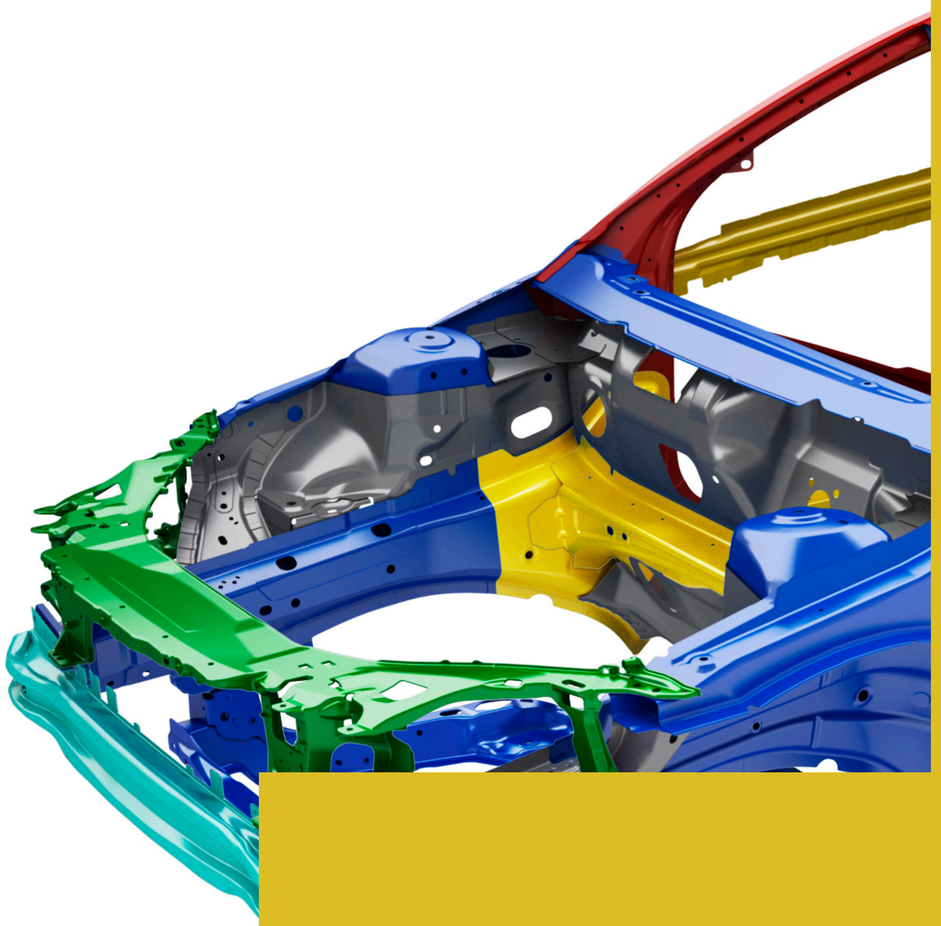
Vibration and waves on flexible structures

- Free vibration of homogeneous beam



Impact on flexible structures

- Impact on bars and beams
- Free vibration of beams
- Effect of boundary conditions/supports
- Single degree-of-freedom (s.d.o.f.) approximation
- Transverse impact on elastic beams





Side impact bars

E. Spisak, 2014



Leg bone fracture

Rubin Institute for Advanced Orthopaedics



Springboard diving at the 2012 Olympics in London

Photo by C. Rose / Getty Images



Drop-weight impact on a beam

T. Zhan et al., *Engineering Failure Analysis* 56:233-243, 2015



UA175 flies toward the
WTC South Tower

Photo by S. Adair / Reuters

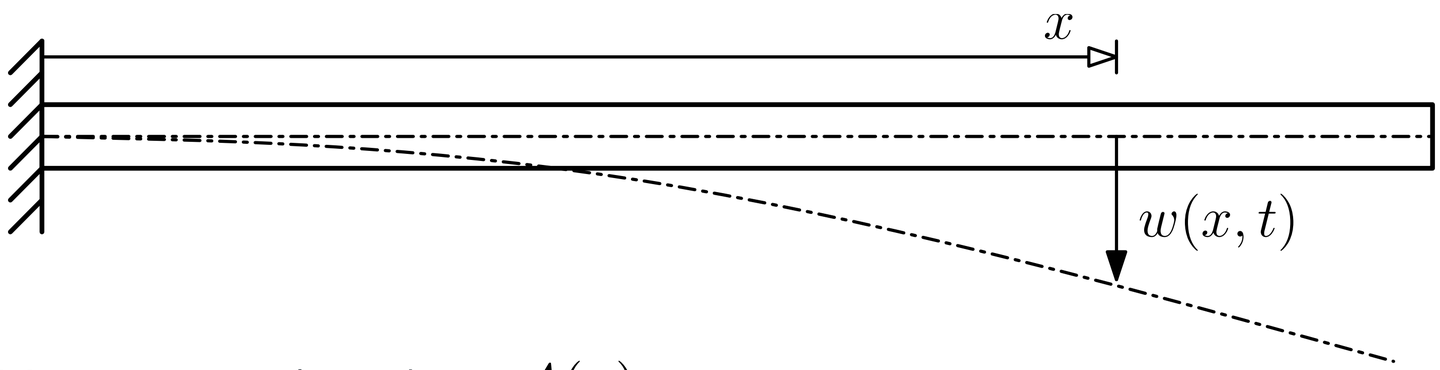
Structural damage on the World Trade Center

Photo by R. Drew / AP Photo



Impact on flexible structures

- Free vibration of homogeneous beam



Mass per unit length: $\rho A(x)$

Bending stiffness: $EI(x)$

Impact on flexible structures

Equation of motion

$$-\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = \rho A\frac{\partial^2 w}{\partial t^2}$$

Or, if the beam is uniform and homogeneous, $0 = \frac{\partial^2 w}{\partial t^2} + \gamma^2\frac{\partial^4 w}{\partial x^4}$ $\left(\gamma^2 = \frac{EI}{\rho A}\right)$

with a standing wave solution (for single mode shape): $w(x,t) = X(x)\left(A_1\cos\omega_0 t + A_2\sin\omega_0 t\right)$

Impact on flexible structures

Standing wave approximation:

$$w(x,t) = X(x)\left(A_1 \cos\omega_0 t + A_2 \sin\omega_0 t\right)$$

$$\Rightarrow \quad 0 = \frac{\mathrm{d}^4 X}{\mathrm{d}x^4} - \frac{\omega_0^2}{\gamma^2} X$$

with solution:

$$X(x) = C_1(\cos kx + \cosh kx) + C_2(\cos kx - \cosh kx)$$
$$+C_3(\sin kx + \sinh kx) + C_4(\sin kx - \sinh kx)$$

Impact on flexible structures

Shape function:

$$X(x) = C_1(\cos kx + \cosh kx) + C_2(\cos kx - \cosh kx)$$

Symmetric

$$+C_3(\sin kx + \sinh kx) + C_4(\sin kx - \sinh kx)$$

Anti-symmetric

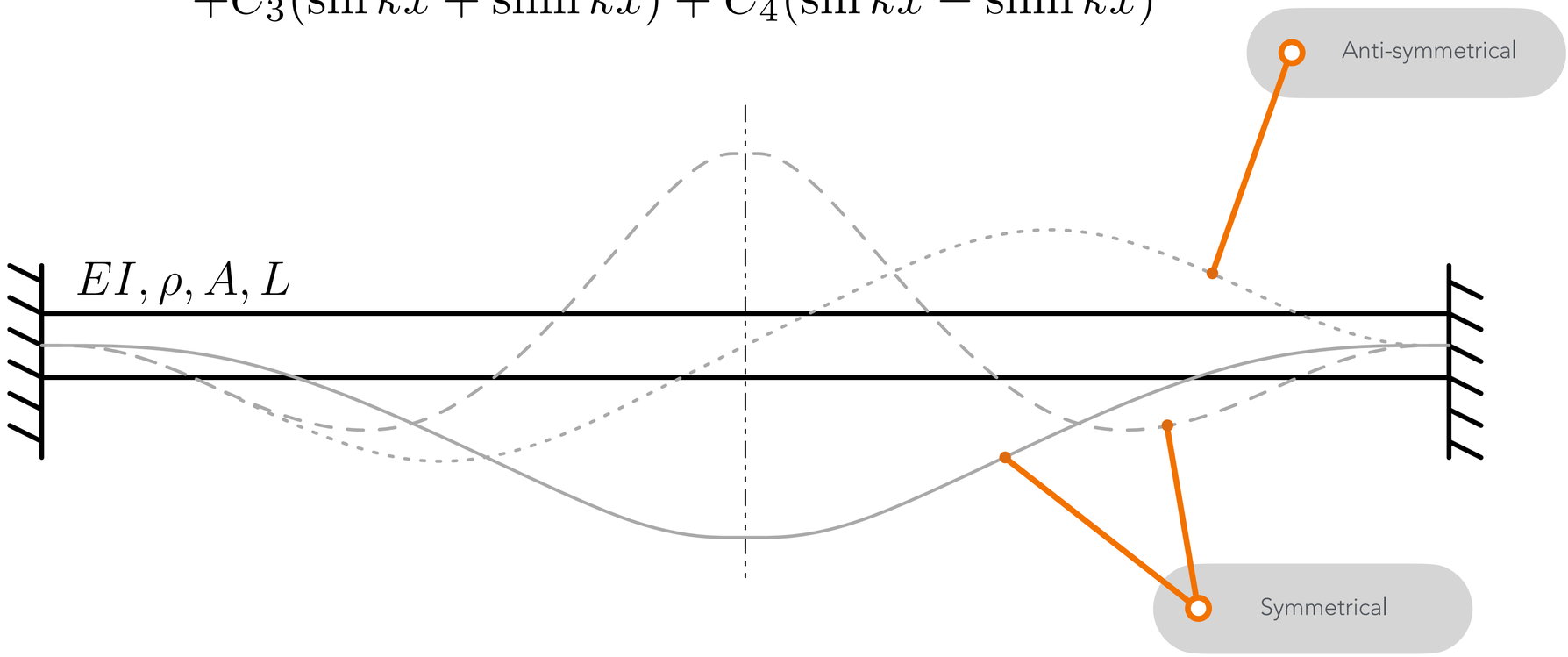
with $k = \sqrt{\frac{\omega_0}{\gamma}}$

Impact on flexible structures

Shape function:

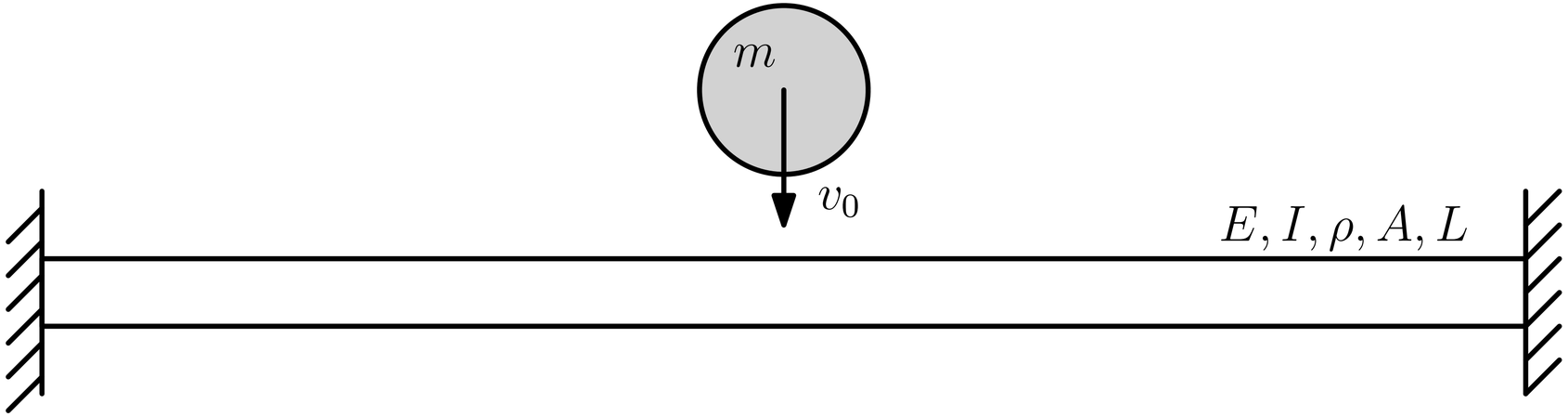
$$X(x) = C_1(\cos kx + \cosh kx) + C_2(\cos kx - \cosh kx)$$

$$+C_3(\sin kx + \sinh kx) + C_4(\sin kx - \sinh kx)$$



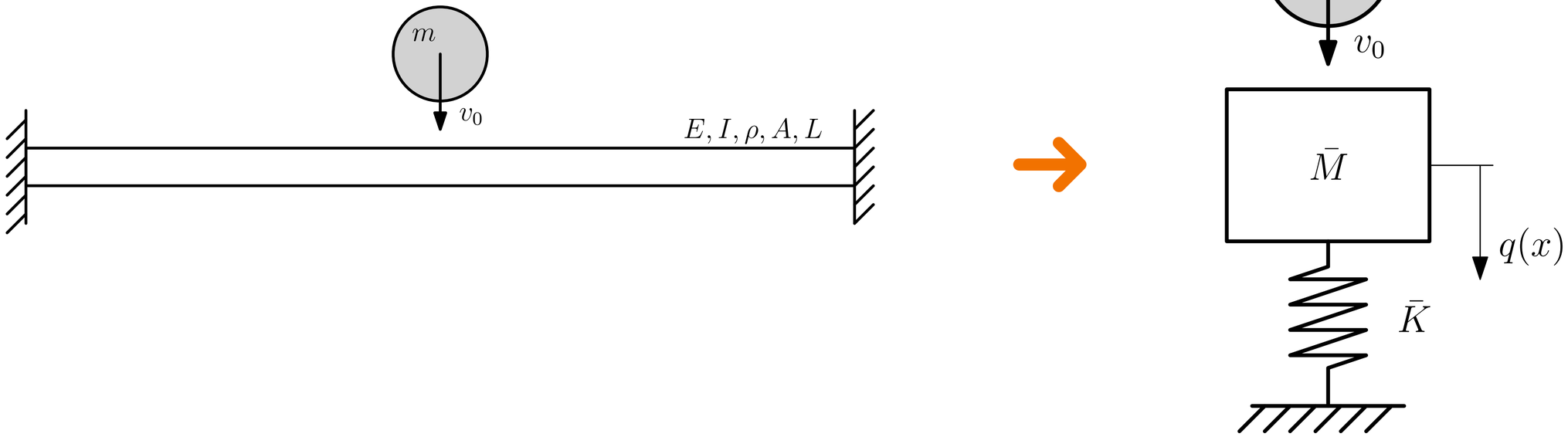
Single degree of freedom approximation

- The main idea is to approximate



Single degree of freedom approximation

- The main idea is to approximate



Impact on flexible structures

Boundary conditions:

$$w(0,t) = w(L,t) = 0 \quad \Rightarrow \quad X(0) = X(L) = 0$$

$$\frac{\partial w}{\partial x}(0,t) = \frac{\partial w}{\partial x}(L,t) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}X}{\partial x}(0) = \frac{\mathrm{d}X}{\partial x}(L) = 0$$

$$\Rightarrow \quad C_1 = C_3 = 0$$

$$\Rightarrow \quad \frac{C_2}{C_4} = \frac{\cos kL - \cosh kL}{\sin kL + \sinh kL} = -\frac{\sin kL - \sinh kL}{\cos kL - \cosh kL}$$

$$\cos kL \cosh kL = 1 \quad \Rightarrow \quad k = \dots$$

Rayleigh-Ritz approximation

- When/if more modes are required:

Deflection function

$$w(x,t) = \sum_{j=0}^{\infty} A_j X_j(x) \sin(\omega_j t - \phi_j)$$

Amplitude — Shape function — Phase angle

- Kinetic energy:

$$T_j = \frac{1}{2}\int_0^L \rho A \dot{w}_j^2(x,t)\mathrm{d}x = \frac{[A_j\omega_j\cos(\omega_j t - \phi_j)]^2}{2}\int_0^L \rho A X_j^2(x)\mathrm{d}x$$

Rayleigh-Ritz approximation

- Kinetic energy:

$$T_j = \frac{1}{2}\int_0^L \rho A \dot{w}_j^2(x,t)\mathrm{d}x = \frac{\left[A_j\omega_j\cos(\omega_j t - \phi_j)\right]^2}{2}\int_0^L \rho A X_j^2(x)\mathrm{d}x$$

Maximum kinetic energy → $$T_j|_{\max} = \frac{A_j^2\omega_j^2}{2}\int_0^L \rho A X_j^2(x)\mathrm{d}x$$

- Potential energy:

$$U_j = \frac{1}{2}\int_0^L \mathcal{M}(x,t)\frac{\mathrm{d}^2 w_j(x,t)}{\mathrm{d}x^2}\mathrm{d}x = \frac{1}{2}\int_0^L EI\left[\frac{\mathrm{d}^2 w_j(x,t)}{\mathrm{d}x^2}\right]^2\mathrm{d}x$$

Maximum potential energy → $$U_j|_{\max} = \frac{A_j^2}{2}\int_0^L EI\left(\frac{\mathrm{d}^2 X_j}{\mathrm{d}x^2}\right)^2\mathrm{d}x$$

Rayleigh-Ritz approximation

Maximum kinetic energy

$$T_j|_{\max} = \frac{A_j^2 \omega_j^2}{2} \int_0^L \rho A X_j^2(x) \mathrm{d}x$$

Maximum potential energy

$$U_j|_{\max} = \frac{A_j^2}{2} \int_0^L EI \left(\frac{\mathrm{d}^2 X_j}{\mathrm{d}x^2} \right)^2 \mathrm{d}x$$

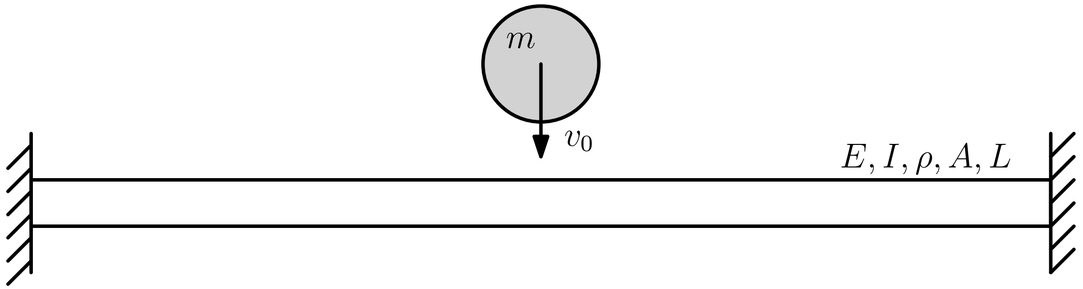
Conservative system: $U_j|_{\max} = T_j|_{\max}$

Rayleigh-Ritz approximation

Conservative system: $U_j|_{\max} = T_j|_{\max}$

$$\omega_j^2 = \int_0^L EI \left(\frac{\mathrm{d}^2 X_j}{\mathrm{d}x^2} \right)^2 \mathrm{d}x \Big/ \int_0^L \rho A X_j^2(x) \mathrm{d}x$$

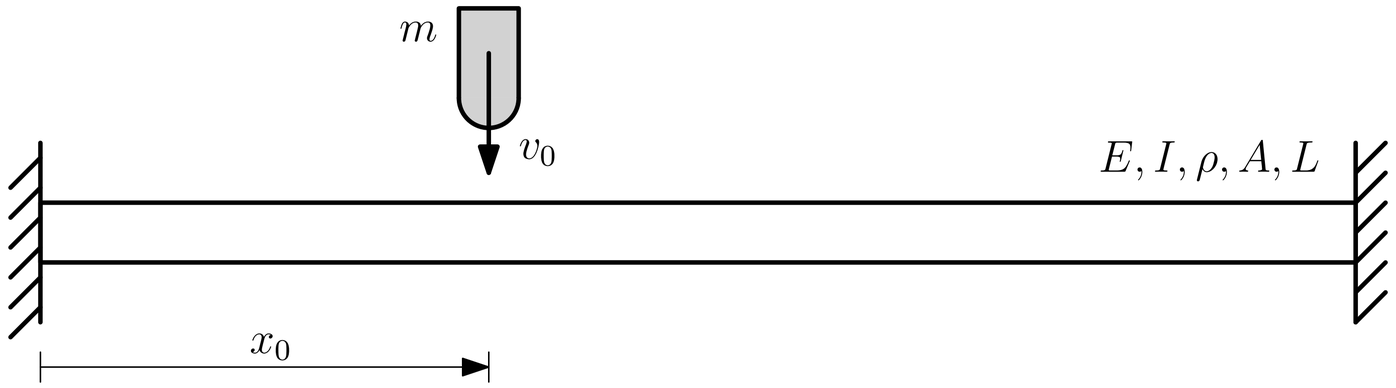
Transverse impact on elastic beams



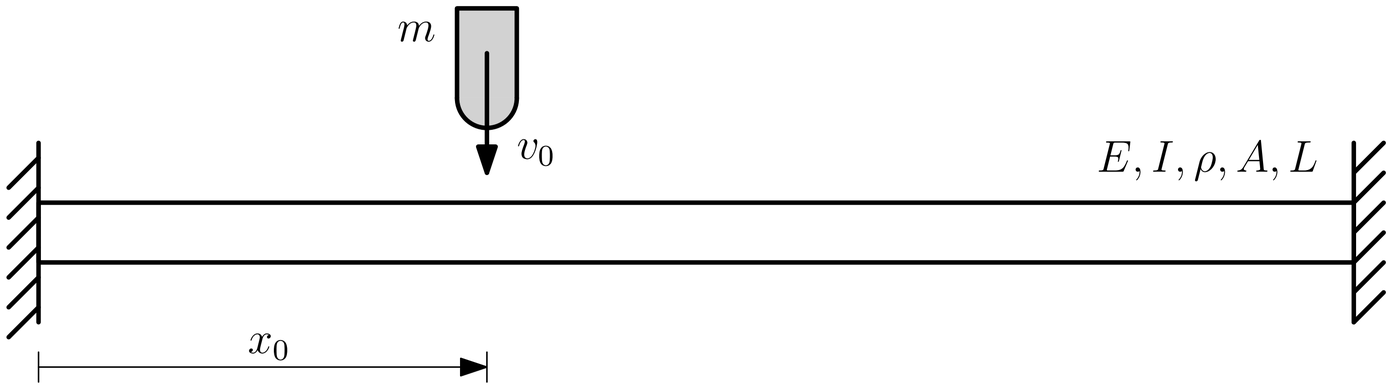
- What is relevant:
  - Mass ratio: $m/(\rho AL)$
  - Location of impact relative to boundaries (and BC)
  - Deformability of impacted area (compared to structural compliance)
- Different methods for different factors

Impact of plastic *missile*

- Orthogonal impact
- Perfectly plastic missile
- Local deformation (indentation) on beam is negligible



Impact of plastic *missile*



- Contact force:

$$F(t) = -m\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$F(t)\left(= -ma = -m\,\ddot{\omega}|_{x_0}\right) = -m|X(x_0)|\frac{\mathrm{d}^2q}{\mathrm{d}t^2}$$

Impact of plastic *missile*

$$F(t)\left(=-ma=-m\,\ddot{\omega}|_{x_0}\right)=-m|X(x_0)|\frac{\mathrm{d}^2q}{\mathrm{d}t^2}$$

Consequently,

$$\omega_j^2 q_j+\frac{\mathrm{d}^2q_j}{\mathrm{d}t^2}=\frac{-m}{\rho A}\frac{\mathrm{d}^2q_j}{\mathrm{d}t^2}|X_j(x_0)|\Big/\int_0^L X_j^2(x)\mathrm{d}x$$

$$q_j=A_j\sin\left(\bar{\omega}_j t+\bar{\phi}_j\right)$$

Amplitude

Modified modal frequency (accounts for added mass)

Phase angle

Impact of plastic *missile*

$$F(t)\left(=-ma=-m\,\ddot{\omega}|_{x_0}\right)=-m|X(x_0)|\frac{\mathrm{d}^2q}{\mathrm{d}t^2}$$

Consequently,

$$\omega_j^2 q_j+\frac{\mathrm{d}^2q_j}{\mathrm{d}t^2}=\frac{-m}{\rho A}\frac{\mathrm{d}^2q_j}{\mathrm{d}t^2}|X_j(x_0)|\Big/\int_0^L X_j^2(x)\mathrm{d}x$$

$$q_j=A_j\sin\left(\bar{\omega}_j t+\bar{\phi}_j\right)$$

$$\bar{\omega}_j^2=\omega_j^2\Big/\left(1+\frac{m|X_j(x_0)|}{\rho A\int_0^L X_j^2\mathrm{d}x}\right)$$

Impact of plastic *missile*

- Displacement of modified system:

$$w(x,t) = \sum_{j=1}^{\infty} A_j \sin\left(\bar{\omega}_j t + \bar{\phi}_j\right) X_j(x)$$

- Velocity of modified system:

$$\dot{w}(x,t) = \frac{\mathrm{d}w(x,t)}{\mathrm{d}t} = \sum_{j=1}^{\infty} A_j \bar{\omega}_j \cos\left(\bar{\omega}_j t + \bar{\phi}_j\right) X_j(x)$$

with initial conditions

$$w(x,0) = 0, \quad \dot{w}(x,0) = m v_0 \delta(x_0)$$

Impact of plastic *missile*

- Initial conditions:

$$w(x,0) = 0, \quad \dot{w}(x,0) = mv_0\delta(x_0)$$

$$\Rightarrow 0 = A_j \sin\bar{\phi}_j$$

$$\Rightarrow \int_0^L \dot{w}(x,0)X_j(x)\mathrm{d}x = A_j\bar{\omega}_j \cos\bar{\phi}_j \int_0^L X_j^2\mathrm{d}x$$

$$\bar{\phi}_j = 0, \quad A_j = \frac{mv_0X_j(x_0)}{\bar{\omega}_j\rho A\int_0^L X_j^2\mathrm{d}x}$$

Impact of plastic *missile*

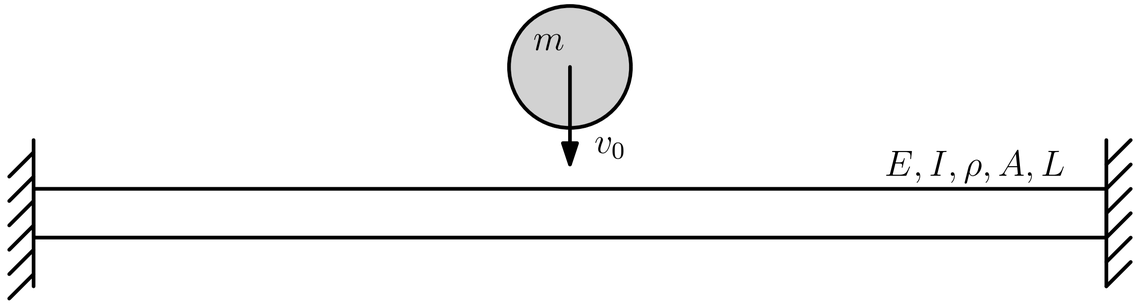
Modified frequency and amplitude of motion:

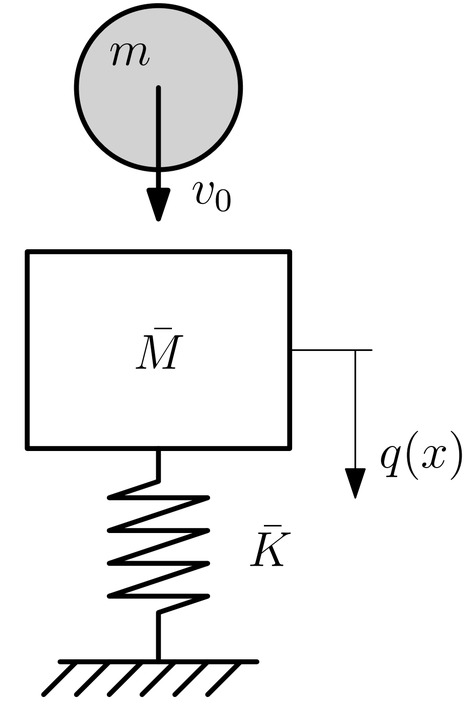
$$\bar{\omega}_j^2 = \omega_j^2 \Big/ \left(1 + \frac{m|X_j(x_0)|}{\rho A \int_0^L X_j^2 \mathrm{d}x}\right)$$

$$\bar{\phi}_j = 0, \quad A_j = \frac{m v_0 X_j(x_0)}{\bar{\omega}_j \rho A \int_0^L X_j^2 \mathrm{d}x}$$

Single degree of freedom approximation

- The main idea is to approximate





Example

Study the response of the beam shown if it is at rest before it is impacted orthogonally at mid-span by a rigid missile.

